Chapter 7 Uncertainty

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Outline

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Stochastic Processes

Properties of Stochastic Processes

A stochastic process $\{X_t : t \in T\}$ is a collection of random variables indexed by time.

- **Conditional expectation**: $\mathbb{E}_t[X_{t+1}]$ conditions on all information available at t
- **Law of iterated expectations**: for $t < s < \tau$,

$$\mathbb{E}_t[\mathbb{E}_s[X_\tau]] = \mathbb{E}_t[X_\tau]$$

- **Covariance stationarity**: unconditional means and autocovariances do not depend on t
- **Ergodicity**: time averages converge to population moments, $(1/T)\sum_{t=1}^{T} X_t \to \mathbb{E}[X_t]$
- **Markov**: $\Pr[X_{t+k} = x | X_\tau, \forall \tau \leq t] = \Pr[X_{t+k} = x | X_t]$
- **Stationary distribution**: $\bar{\pi}(X)$ such that if $X_t \sim \bar{\pi}$, then $X_{t+1} \sim \bar{\pi}$

Markov Chains

x_t takes values in $\mathcal{X} = \{\bar{x}^1, \ldots, \bar{x}^N\}$. Transition matrix P: $P_{ij} = \Pr[x_{t+1} = \bar{x}^j | x_t = \bar{x}^i]$.

Each row sums to one. Given initial distribution π_0 (a $1 \times N$ vector):

$$\pi_{t+k} = \pi_t \times P^k$$

Stationary distribution: $\bar{\pi} = \bar{\pi} \times P$, i.e., $\bar{\pi}'$ is the eigenvector of P' with unit eigenvalue.

Employment/unemployment example: $P = \begin{pmatrix} 1-s & s \\ f & 1-f \end{pmatrix}$

Steady-state unemployment: $\bar{u} = s/(s+f)$. Convergence from any u_0.

Convergence of Markov Chains

Not all Markov chains converge to a unique stationary distribution.

- $P = I$ (identity): any distribution is stationary (not unique)
- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: unique stationary distribution $[1/2, 1/2]$ but distribution oscillates, never converges

Sufficient conditions for convergence and uniqueness: State j is **reachable** from i if $\Pr(x_n = \bar{x}^j | x_0 = \bar{x}^i) > 0$ for some n. **Irreducible**: all states reachable from all others. **Aperiodic**: for each i, $\exists n$ such that $\Pr(x_{n'} = \bar{x}^i | x_n = \bar{x}^i) > 0$ for all $n' \geq n$.

An irreducible chain has a unique $\bar{\pi}$. If also aperiodic, $\lim_{t\to\infty} \pi_0 P^t = \bar{\pi}$ for all π_0.

AR(1) Process

$$x_t = \rho x_{t-1} + b\varepsilon_t + (1-\rho)\mu$$

with $\mathbb{E}_{t-1}[\varepsilon_t] = 0$, $\mathbb{E}_{t-1}[\varepsilon_t^2] = 1$, $\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_{t+s}] = 0$ for $s > 0$.

If $|\rho| < 1$: stationary. Moving-average representation: $x_t = \mu + b\sum_{s=0}^{\infty} \rho^s \varepsilon_{t-s}$.

Moments:

- $\mathbb{E}[x_t] = \mu$
- Var$(x_t) = b^2/(1-\rho^2)$
- Corr$(x_t, x_{t+j}) = \rho^j$

μ controls the level, b the volatility, ρ the persistence.

Vector Linear Stochastic Difference Equations

$x_t \in \mathbb{R}^n$, $\varepsilon_t \in \mathbb{R}^m$ with $\mathbb{E}_t[\varepsilon_{t+1}] = 0$, $\mathbb{E}_t[\varepsilon_{t+1}\varepsilon'_{t+1}] = I$:

$$x_t = Ax_{t-1} + B\varepsilon_t + C$$

- Stationary if all eigenvalues of A are < 1 in absolute value
- Unconditional mean: $\mu = (I - A)^{-1}C$
- Unconditional covariance: $\Gamma(0) = A\Gamma(0)A' + BB'$ (Lyapunov equation)

Impulse response function: Effect of shock ε at t on x_{t+h} (no further shocks):

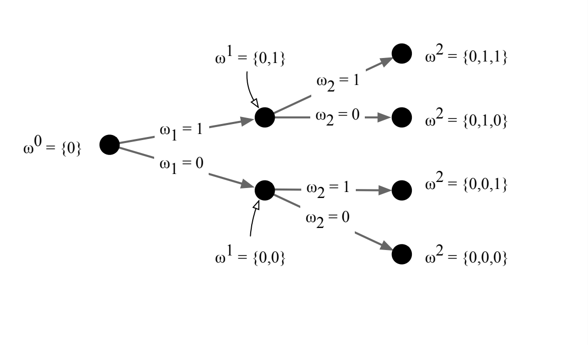
$$x_{t+h} - \mu = A^{h+1}(x_{t-1} - \mu) + A^h B\varepsilon_t$$

$F(h) = A^h B\varepsilon$ is the IRF of x to ε.

Choice Under Uncertainty

Stochastic Events and Event Trees

Example event tree



- $\omega_t \in \Omega_t$: stochastic event at date t
- $\omega^t = \{\omega_0, \omega_1, \dots, \omega_t\} \in \Omega^t$: history of events up to date t
- $\pi_t(\omega^t)$: date-0 probability of history ω^t
- $\pi_t(\omega^t|\omega^\tau)$: conditional probability of ω^t given ω^τ $(t > \tau)$
- Outcomes at t are functions of the full history: e.g., $a_t(\omega^t)$

Expected Utility and Risk Aversion

Consumption process $\{c_t(\omega^t)\}$ ranked by:

$$\sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \pi_t(\omega^t)\beta^t u(c_t(\omega^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Risk aversion: u concave $\Rightarrow$ less risky stream preferred (Jensen's inequality).

Absolute risk aversion: $-u''(c)/u'(c)$.
CARA: $u(c) = -\exp(-\alpha c)$, coefficient $= \alpha$.

Relative risk aversion: $-cu''(c)/u'(c)$.
CRRA: $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, coefficient $= \sigma$.

With CRRA, the EIS $= 1/\sigma$. A single parameter σ governs both risk aversion and intertemporal substitution.

Portfolio Choice: CRRA vs. CARA

Investor allocates W between risk-free (R_f) and risky (Z) assets. Let $A =$ risky investment.

CRRA ($u(c) = c^{1-\sigma}/(1-\sigma)$): FOC becomes

$$\mathbb{E}_0 \left[\left(R_f \left(1 - \frac{A}{W} \right) + Z \frac{A}{W} \right)^{-\sigma} (Z - R_f) \right] = 0$$

Solution for A/W does not depend on W: rich and poor choose the **same risky share**.

CARA ($u(c) = -\exp(-\alpha c)$): FOC becomes

$$\mathbb{E}_0 \left[\exp\left(-\alpha(Z - R_f)A \right) (Z - R_f) \right] = 0$$

Solution for A does not depend on W: rich have a **lower risky share**.

Data: rich hold higher risky share $\Rightarrow$ neither model fully matches, but CRRA is closer.

The Stochastic Growth Model

Two-Period Economy: Setup

Period 0: TFP known. Period 1: TFP $A_1(\omega_1)$ is random with probability $\pi_1(\omega_1)$.

Preferences:

$$U = u(C_0) + \beta \sum_{\omega_1 \in \Omega_1} \pi_1(\omega_1) u(C_1(\omega_1))$$

Resource constraints:

$$K_1 + C_0 = K_0^\alpha + (1-\delta)K_0$$

$$C_1(\omega_1) = A_1(\omega_1)K_1^\alpha + (1-\delta)K_1 \qquad \forall \omega_1$$

The planner chooses C_0, K_1, and $\{C_1(\omega_1) : \forall \omega_1\}$—a **contingent plan**.

Two-Period Economy: Euler Equation

Lagrangian with multipliers λ_0 and $\lambda_1(\omega_1)$. FOCs:

$$u'(C_0) = \lambda_0$$

$$\lambda_0 = \sum_{\omega_1 \in \Omega_1} \lambda_1(\omega_1)(\alpha A_1(\omega_1) K_1^{\alpha-1} + 1 - \delta)$$

$$\beta \pi_1(\omega_1) u'(C_1(\omega_1)) = \lambda_1(\omega_1) \qquad \forall \omega_1$$

Combining:

$$u'(C_0) = \beta \sum_{\omega_1 \in \Omega_1} \pi_1(\omega_1) u'(C_1(\omega_1))(\alpha A_1(\omega_1) K_1^{\alpha-1} + 1 - \delta)$$

Stochastic Euler equation: marginal utility loss from saving = expected marginal utility gain, weighted by state-contingent returns.

Infinite-Horizon Economy: Preferences and Constraints

Preferences:

$$U = \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \pi_t(\omega^t) \beta^t u(C_t(\omega^t))$$

Production: $Y_t(\omega^t) = A_t(\omega^t) F(K_t(\omega^{t-1}), L_t(\omega^t))$

Note: K_t depends on ω^{t-1} (chosen before ω_t realized).

Resource constraint:

$$K_{t+1}(\omega^t) + C_t(\omega^t) = f(A_t(\omega^t), K_t(\omega^{t-1}))$$

where $f(A, K) \equiv AF(K, 1) + (1 - \delta)K$, for all t and ω^t.

Infinite-Horizon Euler Equation

FOC for $K_{t+1}(\omega^t)$ (affects production at $t+1$ for all ω^{t+1} extending ω^t):

$$\lambda_t(\omega^t) = \sum_{\{\omega^{t+1}|\omega^t\}} \lambda_{t+1}(\omega^{t+1}) f_2(A_{t+1}(\omega^{t+1}), K_{t+1}(\omega^t))$$

FOC for $C_t(\omega^t)$: $\pi_t(\omega^t)\beta^t u'(C_t(\omega^t)) = \lambda_t(\omega^t)$

Euler equation:

$$u'(C_t(\omega^t)) = \beta \sum_{\{\omega^{t+1}|\omega^t\}} \pi_{t+1}(\omega^{t+1}|\omega^t) u'(C_{t+1}(\omega^{t+1})) f_2(A_{t+1}(\omega^{t+1}), K_{t+1}(\omega^t))$$

Compact notation:

$$u'(C_t) = \beta \mathbb{E}_t[u'(C_{t+1}) f_2(A_{t+1}, K_{t+1})]$$

Recursive Formulation

Assume A follows a first-order Markov process with $\pi(A'|A)$.

Bellman equation:

$$V(A,K) = \max_{C,K' \geq 0} \left\{ u(C) + \beta \sum_{A'} \pi(A'|A) V(A',K') \right\}$$

subject to $K' + C = f(A,K)$.

FOC: $u'(C) = \beta \sum_{A'} \pi(A'|A) V_2(A',K')$

Envelope: $V_2(A,K) = u'(C) f_2(A,K)$

Functional Euler equation (with $K' = g(A,K)$):

$$\begin{aligned} &u'(f(A,K) - g(A,K)) = \\ &\quad \beta \textstyle\sum_{A'} \pi(A'|A) u'(f(A', g(A,K)) - g(A', g(A,K))) f_2(A', g(A,K)) \end{aligned}$$

Must hold for all (A,K).

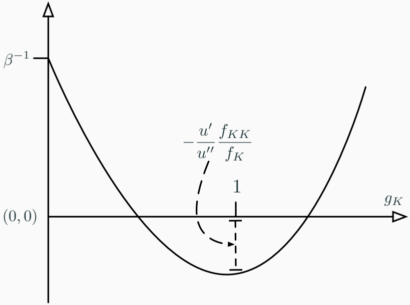
Solving via Linearization

$A' = \rho A + (1-\rho)\bar{A} + \varepsilon'$, $\mathbb{E}[\varepsilon'] = 0$. Linearize around deterministic steady state $(\bar{A}, \bar{K})$. Let $\hat{X}_t \equiv X_t - \bar{X}$.

Linearize the functional Euler equation and matching coefficients on $\hat{K}$ yields a quadratic equation in the coefficient g_K:

$$g_K^2 - \left(1 + \beta^{-1} + \frac{u'}{u''}\frac{f_{KK}}{f_K}\right) g_K + \beta^{-1} = 0$$

Quadratic equation to determine g_K



Linearized Solution

The linearized policy rule is:

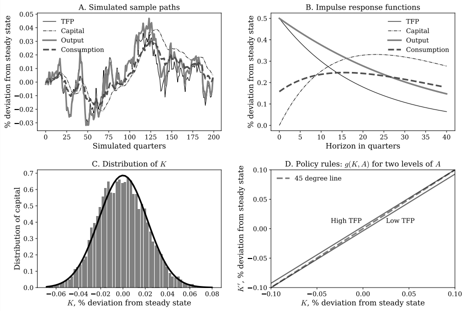
$$K' = \bar{K} + g_K(K - \bar{K}) + g_A(A - \bar{A})$$

g_A can be solved similarly to g_K, using the linearized functional Euler equation. Together with $A' = \bar{A} + \rho(A - \bar{A}) + \varepsilon'$: a system of two linear stochastic difference equations, analyzed using the tools of Section 7.1.

Certainty equivalence: After linearization, only $\mathbb{E}[\varepsilon']$ matters, not Var$[\varepsilon']$. It is as if there is no uncertainty and A' equals its conditional expectation.

Numerical Illustration

Parameters: $f(A,K) = AK^\alpha + (1-\delta)K$, $\alpha = 0.3$, $\delta = 0.02$, $\rho = 0.95$, SD$(\varepsilon) = 0.005$, $\bar{A} = 1$, $u = \log$, $\beta = 0.99$.



- TFP and Y: high-frequency variation. K and C: smooth
- On impact: Y jumps with A (K predetermined). C increases less than Y (saving)
- Policy rules shift with A; K fluctuates between their intersections with the 45-degree line

Competitive Equilibrium Under Uncertainty

Complete vs. Incomplete Markets

Complete markets: For each (t, ω^t), a contract delivers one unit of good at that date-history and zero otherwise (**Arrow securities**). Any risk can be insured.

Incomplete markets: Some contracts unavailable. Some risks cannot be insured.

Setup: Households $i \in I$ with expected utility $\sum_{t=0}^{\infty} \sum_{\omega^t} \pi_t(\omega^t) \beta^t u(c_{i,t}(\omega^t))$. Stochastic endowments $y_{i,t}(\omega^t)$. Arrow-Debreu prices $p_t(\omega^t)$.

Definition 13 (AD CE Under Uncertainty)

An AD CE is $\{c^*_{i,t}(\omega^t) : \forall t, \omega^t\}$ for each $i \in I$, and $\{p_t(\omega^t) : \forall t, \omega^t\}$ such that

1. for each i, $\{c^*_{i,t}(\omega^t)\}$ solves

$$\max_{\{c_t(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi_t(\omega^t) u(c_t(\omega^t))$$

   subject to

$$\sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} p_t(\omega^t) c_t(\omega^t) = \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} p_t(\omega^t) y_{i,t}(\omega^t)$$

2. $\sum_{i \in I} c^*_{i,t}(\omega^t) = \sum_{i \in I} y_{i,t}(\omega^t)$ for all (t, ω^t)

Characterization: Insurance and Risk Sharing

FOC: $\beta^t \pi_t(\omega^t) u'(c_{i,t}(\omega^t)) = \lambda_i p_t(\omega^t)$

Insurance: If prices are actuarially fair $\left(p_t(\omega^t) = \bar{p}_t \pi_t(\omega^t)\right)$, then $u'(c_{i,t}(\omega^t)) = u'(c_{i,t}((\omega^t)'))$ for all $\omega^t, (\omega^t)' \Rightarrow$ **full insurance**: $c_{i,t}$ independent of ω^t.

Risk sharing: Take the ratio of FOCs for households i and j:

$$\frac{u'(c_{i,t}(\omega^t))}{u'(c_{j,t}(\omega^t))} = \frac{\lambda_i}{\lambda_j}$$

λ_i/λ_j is constant across time and states $\Rightarrow c_{i,t}$ depends only on **aggregate** endowment, not on $y_{i,t}$. All idiosyncratic risk is insured away.

Sequential Trading: Arrow Securities

For each ω_{t+1}, an asset traded at t pays 1 unit at $t+1$ if ω_{t+1} occurs. Price: $q_t(\omega_{t+1}|\omega^t)$.

Budget constraint:

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega_{t+1}) \leq y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

nPg: $a_{i,t}(\omega^t) \geq -\sum_{\tau=t}^{\infty} \sum_{\omega^\tau} \tilde{q}_\tau^t(\omega^\tau) y_{i,\tau}(\omega^\tau)$

where $\tilde{q}_{\tau+1}^t(\omega^{\tau+1}) = q_\tau(\omega_{\tau+1}|\omega^\tau)\tilde{q}_\tau^t(\omega^\tau)$ with $\tilde{q}_t^t(\omega^t) = 1$.

Definition 14 (Sequential CE Under Uncertainty)

A sequential CE is $\{c^*_{i,t}(\omega^t)\}$, $\{a^*_{i,t+1}(\omega^t)\}$ for each i, and $\{q_t(\omega^t)\}$ such that

1. for each i, $\{c^*_{i,t}(\omega^t)\}$ and $\{a^*_{i,t+1}(\omega^t)\}$ solve

$$\max_{\{c_t(\omega^t), a_{t+1}(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi(\omega^t) u(c_t(\omega^t))$$

   subject to

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega_{t+1}) = y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

   and the nPg condition, with $a_{i,0} = 0$

2. $\sum_i (c^*_{i,t}(\omega^t) - y_{i,t}(\omega^t)) = 0$ and $\sum_i a^*_{i,t+1}(\omega^t) = 0$ for all (t, ω^t)

The sequential and AD equilibria deliver the same allocations (same consolidation argument as Chapter 5).

Spanning and Complete Markets

S states, N assets. Payoff matrix D $(S \times N)$. Portfolio $\theta \in \mathbb{R}^N$. Payoff vector: $D\theta$.

Market span: $\mathcal{M} \equiv \{z \in \mathbb{R}^S : z = D\theta \text{ for some } \theta \in \mathbb{R}^N\}$

Markets are complete if $\mathcal{M} = \mathbb{R}^S$, i.e., $\text{rank}(D) = S$.

If complete: $\exists$ portfolios $\{\theta_j^A\}_{j=1}^S$ such that $D\theta_j^A = e_j$ (Arrow securities).

Real-world assets pay off in multiple states, but completeness only requires that Arrow securities can be **replicated** by portfolios.

Competitive Equilibrium in the Stochastic Growth Model

Setup

Representative household with k_0 units of capital, one unit of labor (inelastic).

Production:

$$y_t(\omega^t) = A_t(\omega^t)F(k_t(\omega^{t-1}), 1)$$

Factor prices:

$$r_t(\omega^t) = A_t(\omega^t)F_1(k_t(\omega^{t-1}), 1),$$
$$w_t(\omega^t) = A_t(\omega^t)F_2(k_t(\omega^{t-1}), 1)$$

Budget:

$$c_t(\omega^t) + k_{t+1}(\omega^t) = (r_t(\omega^t) + 1 - \delta)k_t(\omega^{t-1}) + w_t(\omega^t)$$

Household Euler equation:

$$u'(c_t(\omega^t)) = \mathbb{E}_t[\beta u'(c_{t+1}(\omega^{t+1}))(r_{t+1}(\omega^{t+1}) + 1 - \delta)]$$

Definition 15 (Sequential CE, Stochastic Growth Model)

A sequential CE is $\{c_t^*(\omega^t)\}$, $\{k_{t+1}^*(\omega^t)\}$, $\{r_t(\omega^t), w_t(\omega^t)\}$ for all (t, ω^t) such that

1. $\{c_t^*(\omega^t)\}$ and $\{k_{t+1}^*(\omega^t)\}$ solve

$$\max_{\{c_t(\omega^t), k_{t+1}(\omega^t)\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \pi(\omega^t) u(c_t(\omega^t))$$

   subject to $c_t(\omega^t) + k_{t+1}(\omega^t) = (r_t(\omega^t) + 1 - \delta) k_t(\omega^{t-1}) + w_t(\omega^t)$

   and the nPg condition

2. $r_t(\omega^t) = A_t(\omega^t) F_1(k_t^*(\omega^{t-1}), 1)$ and $w_t(\omega^t) = A_t(\omega^t) F_2(k_t^*(\omega^{t-1}), 1)$
3. $k_{t+1}^*(\omega^t) + c_t^*(\omega^t) = (1-\delta) k_t^*(\omega^{t-1}) + A_t(\omega^t) F(k_t^*(\omega^{t-1}), 1)$

Condition 3 is redundant (Walras' Law with CRS). The CE allocation coincides with the planner's solution; welfare theorems apply with goods indexed by (t, ω^t).

Incomplete Markets

Incomplete-Market Consumption-Saving Problem

Consumer with stochastic income y_t (Markov), single asset with gross return $1 + r$:

$$V(a, y) = \max_{a' \geq \underline{a}} \left\{ u(a + y - qa') + \beta \mathbb{E}[V(a', y')|y] \right\}$$

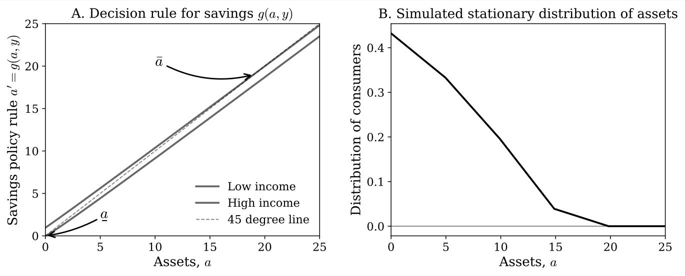
where $q = 1/(1 + r)$.

Euler equation: $u'(c) = (1 + r)\beta \mathbb{E}[u'(c')|y]$

Single asset cannot insure against income risk $\Rightarrow$ consumption fluctuates with y_t.

Partial self-insurance: Accumulate savings to spend down during low-income periods. Fully smooth consumption is not optimal (would require consuming at the worst-case level forever).

Decision Rules and Stationary Distribution



- High income: accumulate savings up to $\bar{a}$ (upper line crosses 45-degree)
- Low income: spend down savings to $\underline{a}$ (borrowing constraint)
- Ergodic set: $[\underline{a}, \bar{a}]$; within it, assets drift up or down
- Stationary distribution: reflects accumulated luck/misfortune across consumers
- Contrast with complete markets: in the deterministic steady state of Chapter 5, $g(a) = a$ (45-degree line)

Summary

Key Takeaways (I)

1. **Stochastic processes**: Markov chains ($\pi_{t+k} = \pi_t P^k$), AR(1) (ρ persistence, b volatility), vector LSDE ($x_t = Ax_{t-1} + B\varepsilon_t + C$, IRF $= A^h B\varepsilon$)
2. **Expected utility**: $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$; risk aversion from concavity; CRRA ties together RRA and EIS via σ
3. **Portfolio choice**: CRRA $\Rightarrow$ constant risky share; CARA $\Rightarrow$ constant risky amount
4. **Stochastic Euler equation**: $u'(C_t) = \beta \mathbb{E}_t[u'(C_{t+1}) f_2(A_{t+1}, K_{t+1})]$; returns weighted by state-contingent marginal utilities

Key Takeaways (II)

5. **Linearization**: Quadratic in g_K (one stable root < 1); certainty equivalence (only $\mathbb{E}[\varepsilon']$ matters, not Var$[\varepsilon']$)
6. **Complete markets** (Def 13–14): Full risk sharing; idiosyncratic risk insured; $c_{i,t}$ depends only on aggregate endowment. Markets complete iff rank$(D) = S$.
7. **Stochastic CE of NGM** (Def 15): Same as planner's solution; welfare theorems hold with goods indexed by (t, ω^t)
8. **Incomplete markets**: Single asset, Euler inequality at borrowing constraint; partial self-insurance; stationary wealth distribution emerges from accumulated idiosyncratic shocks